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Dissipative dynamics for quantum spin systems on a lattice

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Abstract. We show that for a large class of interactions there exist translation-invariant dissipative dynamics which satisfy the detailed balance condition (in the associated noncommutative symmetric $\mathbb{L}_2$ space), which do not commute with Hamiltonian dynamics and have exponential decay to equilibrium in the high-temperature region.

1. Introduction

The main problem in the domain of the nonequilibrium description of quantum systems is how to construct a translation-invariant semigroup on a noncommutative algebra which not only preserves the unit and positivity in the algebra, but also satisfies a detailed balance condition (that is the self-adjointness of the dynamics in an appropriate Hilbert space). In [6–9] we have shown that it is useful to employ noncommutative $\mathbb{L}_p$ spaces to study stochastic dynamics satisfying a detailed balance condition in some appropriately chosen $\mathbb{L}_2$ space associated to a given Gibbs state. In particular we have argued that to define generators of stochastic dynamics of jump type one should use generalized conditional expectations. Such generalized conditional expectations have been introduced first for finite systems in [1], and then in the general setting of von Neumann algebras in [2]. In our works [6–9] we have given a different construction of them from the point of view of $\mathbb{L}_2$ spaces. We have also shown that one can use these expectations for an infinite system to define explicitly the quantum analogues of Glauber and Kawasaki dynamics which satisfy a detailed balance condition in a suitable noncommutative $\mathbb{L}_2$ space. Moreover, we have formulated general sufficient conditions for the existence and ergodicity of translation-invariant dynamics, similar to the one used in the classical case. In section 2 we show that one can extend these results to include the Hamiltonian term in the generator (which is absent in the case of classical discrete spin systems). We also observe there that as long as the potential used to define this Hamiltonian term is sufficiently small, the system remains ergodic. Our general construction is done in the framework of von Neumann algebra associated to a given Gibbs state and thus the corresponding dynamics lives on the von Neumann algebra. To study ergodic properties of the dynamics it would be natural to define it on the $\boldsymbol{C}^*$ inductive limit algebra, which could be regarded as an analogue of the space continuous functions used for the description of classical spin systems. (In this context we can talk about the Feller property of the semigroup.) In general we do not know whether or not it is possible. However, in section 3 we show that it is true for a large class of systems. (Our result in some sense complements those by [5] and [10, 11], where semigroups in the ground-state representation have been considered which—when restricted to classical local observables—possessed the Feller property.) The symmetric parts of the generators of our semigroups satisfy the detailed balance condition in a $\mathbb{L}_2$ space associated to a Gibbs state defined by a quasiclassical interaction of finite range. Thus, we also know in advance a large class of equilibrium states.

Given a Hamiltonian automorphism group, one can always define a family of subordinated dissipative semigroups (which do have the Feller property). By construction they commute with the original Hamiltonian dynamics. In general it is an indication that such semigroups do not have strong ergodic properties (since using them it is impossible to distinguish the states at a given temperature). In this paper we show (see the appendix) that our dissipative dynamics introduced in section 3 do not commute with the Hamiltonian dynamics. (One can expect that a similar result is true for general quantum Glauber dynamics considered in [6–9], as one can easily prove this property for a finite volume block spin-flip dynamics.)

In section 4 we show that the quantum dynamics of block spin-flip type introduced in section 3 for large blocks is strongly ergodic on the inductive limit C^* algebra in the large high-temperature domain. Using this information we also show that the generator for any other block spin-flip dynamics for the same potential has a spectral gap and thus the corresponding dynamics is exponentially ergodic in $\mathbb{L}_2$ sense. By this we extend a classical result of [3] to a noncommutative situation.

In the remainder of this section we recall the basic notations used in the description of quantum lattice systems. The basic role in this description is played by a C^*-algebra $\mathcal{A}$, with norm $||\cdot||$, defined as the inductive limit over a finite-dimensional complex matrix algebra $\mathbf{M}$. In analogy with the classical commutative spin systems, it is natural to view $\mathcal{A}$ as a noncommutative analogue of the space of bounded continuous functions. To every finite set X of the lattice $\mathbb{Z}^d$ (which is denoted later by $X \subset\subset \mathbb{Z}^d$), we associate a subalgebra $\mathcal{A}_X$ of operators localized in the set X. For arbitrary subset $\Lambda \subset \mathbb{Z}^d$ one defines $\mathcal{A}_\Lambda$ to be the smallest (closed) subalgebra of $\mathcal{A}$ containing $\bigcup\{\mathcal{A}_X : X \subset\subset \mathbb{Z}^d,\ X \subset \Lambda\}$. An operator $f \in \mathcal{A}$ will be called local if there is some $Y \subset\subset \mathbb{Z}^d$ such that $f \in \mathcal{A}_Y$. The subset of $\mathcal{A}$ consisting of all local operators will be denoted by $\mathcal{A}_0$.

Together with the algebra $\mathcal{A}$ we are given family Tr_X, $X \subset\subset \mathbb{Z}^d$, of *normalized partial traces* on $\mathcal{A}$. We recall that the partial traces Tr_X all have natural properties of *classical conditional expectations*, that is they are (completely) positive, unit-preserving projections defined on the algebra $\mathcal{A}$. Moreover, the family $\{\mathrm{Tr}_X : X \subset\subset \mathbb{Z}^d\}$ is compatible in a similar sense as conditional expectations and one can see that there is a unique state Tr on $\mathcal{A}$, called *the normalized trace*, such that

$$\mathrm{Tr}(\mathrm{Tr}_X\, f) = \mathrm{Tr}(f)$$

for every $X \subset\subset \mathbb{Z}^d$, i.e. the normalized trace can be regarded as a (free) Gibbs state in the similar sense as in classical statistical mechanics.

A system with interaction is described using a notion of an interaction potential, i.e. a family $\Phi \equiv \{\Phi_X \in \mathcal{A}_X\}_{X\subset\subset\mathbb{Z}^d}$ of self-adjoint operators. A Banach space of potentials satisfying

$$||\Phi||_n \equiv \sup_{i\in\mathbb{Z}^d} \sum_{\substack{X\subset\subset\mathbb{Z}^d \\ X\ni i}} |X|^{n-1}||\Phi_X|| < \infty$$

will be denoted by $\mathbb{B}_n$. The potentials in $\mathbb{B}_1$ will be called Gibbsian. A potential $\Phi \equiv \{\Phi_X\}_{X\subset\subset\mathbb{Z}^d}$ is of *finite range* $R \geqslant 0$, iff $\Phi_X = 0$ for all $X \in \mathcal{F}, \mathrm{diam}(X) > R$.

The corresponding Hamiltonian H_Λ and the interaction energy U_Λ in $\Lambda \subset\subset \mathbb{Z}^d$ is defined respectively by

$$H_\Lambda \equiv H_\Lambda(\Phi) \equiv \sum_{X \subset \Lambda} \Phi_X \qquad U_\Lambda \equiv U_\Lambda(\Phi) \equiv \sum_{X \cap \Lambda \neq \emptyset} \Phi_X.$$

Using the Hamiltonian we introduce a density matrix $\rho_\Lambda \equiv \mathrm{e}^{-\beta H_\Lambda} / \mathrm{Tr}\, \mathrm{e}^{-\beta H_\Lambda}$ with $\beta \in (0, \infty)$, and define a finite volume Gibbs state ω_Λ by

$$\omega_\Lambda(f) \equiv \mathrm{Tr}(\rho_\Lambda f).$$

It is known, see e.g. [4], that for $\beta \in (0, \infty)$ the limit state $\omega \equiv \lim_{\mathcal{F}_0} \omega_\Lambda$ (defined with some exhaustion $\mathcal{F}_0$ of the lattice), exists and is faithful on $\mathcal{A}$. For a quantum spin system, unlike for the classical one, we can also introduce a natural Hamiltonian dynamics defined in a finite volume as the following automorphism group associated to potential Φ

$$\alpha_t^\Lambda(f) \equiv \mathrm{e}^{+\mathrm{i}tH_\Lambda} f \mathrm{e}^{-\mathrm{i}tH_\Lambda}.$$

With this dynamics one has the following KMS condition, see e.g. [4], for the finite volume state ω_Λ

$$\omega_\Lambda(f^* g) = \omega_\Lambda(\alpha_{-\mathrm{i}\beta}^\Lambda(g) f^*).$$

If the potential $\Phi \in \mathbb{B}_2$, then the following limit exists, [12],

$$\alpha_t(f) \equiv \lim_{\mathcal{F}_0} \alpha_t^\Lambda(f)$$

for every $f \in \mathcal{A}_0$, where $\Lambda \to \mathbb{Z}^d$ through a Fisher sequence $\mathcal{F}_0$. The generator of this automorphism group α_t is given on the local elements by

$$\delta_\Phi(f) \equiv \lim_{\mathcal{F}_0} \delta_{\Phi,\Lambda}(f) \equiv \lim_{\mathcal{F}_0} \mathrm{i}[H_\Lambda(\Phi), f]$$

where $[F_1, F_2] \equiv F_1 F_2 - F_2 F_1$ stands for the commutator of two operators F_1 and F_2. The infinite volume state ω is called an (α_t, β)-KMS state. Using this (α_t, β)-KMS state, for $s \in [0, 1]$ we introduce on $\mathcal{A}$ the following family of scalar products

$$\langle f, g \rangle_s \equiv \omega((\alpha_{-\mathrm{i}s\beta/2}(f))^* \alpha_{-\mathrm{i}s\beta/2}(g)).$$

The completion of $\mathcal{A}$ in the corresponding norm will be called the (noncommutative) L_2-space and denoted by $\mathbb{L}_2(\omega, s)$. Later the special role will be played by the space defined with $s = \frac{1}{2}$ which is called the symmetric L_2-space.

By $\mathcal{M}$ we will denote the von Neumann algebra obtained via GNS construction, see e.g. [4], using the state ω. The partial trace Tr_X, for $X \subset\subset \mathbb{Z}^d$, can be naturally extended to this von Neumann algebra. Using it we can introduce the following generalized conditional expectation

$$E_X(\boldsymbol{f}) \equiv \mathrm{Tr}_X(\gamma_X^* \boldsymbol{f} \gamma_X)$$

with some bounded operator $\gamma_X \in \mathcal{M}$. In [9] we have shown that given a state $\omega_{\beta\Phi}$ associated to a sufficiently fast decaying potential and sufficiently high temperature or in one dimension a potential of finite range and arbitrary temperature one finds $\gamma_X \in \mathcal{M}$ such that the corresponding generalized conditional expectation is symmetric in the associated (symmetric) L_2 space (which is isomorphic to the one introduced above). Using the generalized conditional expectation one defines the following elementary bounded Markov generator which will play an essential role later

$$\mathcal{L}_X(\boldsymbol{f}) \equiv E_X(\boldsymbol{f}) - \boldsymbol{f}.$$

2. Dissipative dynamics for infinite quantum spin systems

Using the elementary Markov generator $\mathcal{L}_X$ introduced above we define, for a finite volume $\Lambda \subset\subset \mathbb{Z}^d$, a Markov generator $\mathcal{L}_\Lambda^X$ on the algebra $\mathcal{M}$ as follows

$$\mathcal{L}_\Lambda^X \boldsymbol{f} \equiv \sum_{\boldsymbol{j}\in\Lambda} \mathcal{L}_{X+\boldsymbol{j}} \boldsymbol{f}.$$

The operator $\mathcal{L}_\Lambda^X$ is a bounded Markov generator and therefore one can easily define the associated Markov semigroup $P_t^{\Lambda,X} \equiv e^{t\mathcal{L}_\Lambda^X}$ on $\mathcal{M}$. By construction such semigroups preserve the Gibbs states corresponding to a given potential and temperature. However, one can expect that they have rather poor ergodicity properties if Λ is a finite region. Therefore one would like to construct a semigroup P_t^X which is formally a limit of the semigroups $P_t^{\Lambda,X}$ as $\Lambda \to \mathbb{Z}^d$. In [7] and [8] we have presented an extension of a classical strategy which allows this construction. Under some technical conditions it also gives a strong control of the ergodicity of the semigroup. In [7] we were also interested in the Feller property (that is whether the semigroup preserves the algebra $\mathcal{A}$—the natural analogue of continuous functions), it was natural there to give a description in the framework of the $\boldsymbol{C}^*$-algebraic inductive limit $\mathcal{A}$. The method of this construction, as well as conditions for the ergodicity, are more general and remain valid if one replaces the algebra $\mathcal{A}$ and its norm $||\cdot||$ by the von Neumann algebra $\mathcal{M}$ and the corresponding norm $||\cdot||_\mathcal{M}$, respectively, [8].

In the case of quantum systems it is natural to consider the generators which, besides purely dissipative part, contains also a Hamiltonian part. An extension to this more general situation can be obtained in a similar way and below we formulate the corresponding result. To describe these results we need the following notation. Let

$$\partial_{\boldsymbol{j}} \boldsymbol{f} \equiv \boldsymbol{f} - \mathrm{Tr}_{\boldsymbol{j}}\, \boldsymbol{f}$$

with $\mathrm{Tr}_{\boldsymbol{j}}$ being the partial trace on the von Neumann algebra $\mathcal{M}$ at the point $\boldsymbol{j} \in \mathbb{Z}^d$. We define the following seminorm $|||\cdot|||$ in $\mathcal{M}$

$$|||\boldsymbol{f}||| \equiv \sum_{\boldsymbol{j}\in\mathbb{Z}^d} ||\partial_{\boldsymbol{j}} \boldsymbol{f}||_\mathcal{M}.$$

One can see that the seminorm $|||\cdot|||$ is finite on a (dense) subalgebra $\mathcal{M}_1 \subset \mathcal{M}$ containing $\pi_\omega(\mathcal{A}_0)$ and it only vanishes on the centre $\mathcal{Z}_\omega$ of $\mathcal{M}$. We have the following result.

Theorem. Suppose $\mathcal{L}_{X+\boldsymbol{j}} \equiv \mathrm{Tr}_{X+\boldsymbol{j}}(\boldsymbol{a}_{X+\boldsymbol{j}}^*(\cdot)\boldsymbol{a}_{X+\boldsymbol{j}}) - \mathbb{1}$ is a Markov generator defined with the operators $\boldsymbol{a}_{X+\boldsymbol{j}}$ satisfying the following condition

$$||\partial_{\boldsymbol{i}} \boldsymbol{a}_{X+\boldsymbol{j}}||_\mathcal{M} \leqslant \varepsilon\eta(\boldsymbol{i}-\boldsymbol{j})$$

with some constant $\varepsilon \in (0,\infty)$ and a positive function η such that

$$\eta(\boldsymbol{i}-\boldsymbol{j}) \leqslant (|\boldsymbol{i}-\boldsymbol{j}|+1)^{-(2d+\kappa)}$$

for some positive κ. Define

$$\mathcal{L}^{X,\Lambda} = \sum_{\boldsymbol{j}\in\Lambda} \mathcal{L}_{X+\boldsymbol{j}} + \lambda\cdot\delta_{\Psi,\Lambda}$$

for some potential $\Psi \in \mathbb{B}_2$ and $\lambda \in \mathbb{R}$, and set $\boldsymbol{P}_t^{X,\Lambda} \equiv e^{t\mathcal{L}^{X,\Lambda}}$. Then the infinite volume limit

$$\boldsymbol{P}_t^X \boldsymbol{f} \equiv \lim_{\mathcal{F}_0} \boldsymbol{P}_t^{\Lambda,X} \boldsymbol{f}$$

exists. Moreover there is a constant $\varepsilon_0 \in (0,\infty)$ such that if $\varepsilon \in (0,\varepsilon_0)$, then the semigroup $\boldsymbol{P}_t^X$ is strongly ergodic in the sense that

$$|||\boldsymbol{P}_t^X \boldsymbol{f}||| \leqslant \mathrm{e}^{-mt}|||\boldsymbol{f}|||$$

with some $m \in (0,\infty)$ for every $\boldsymbol{f} \in \mathcal{M}_1$, provided that $|\lambda| < \lambda_0$ for some $\lambda_0 > 0$.

2.1. The idea of the proof of exponential decay to equilibrium

Let $\tilde{\mathcal{L}}^X \equiv \sum_{\boldsymbol{j}\in\mathbb{Z}^d} \mathcal{L}_{X+\boldsymbol{j}}$, i.e. we have $\mathcal{L}^X = \tilde{\mathcal{L}}^X + \lambda\delta_\Psi$. For $\boldsymbol{j} \in \mathbb{Z}^d$, let $\tilde{\mathcal{L}}^{X,\boldsymbol{j}} \equiv \tilde{\mathcal{L}}^X - \sum_{\boldsymbol{k}:X+\boldsymbol{k}\ni\boldsymbol{j}} \mathcal{L}_{X+\boldsymbol{k}}$ and let $\mathcal{L}^{X,\boldsymbol{j}} \equiv \tilde{\mathcal{L}}^{X,\boldsymbol{j}} + \lambda\delta_\Psi$. Setting $\boldsymbol{P}_t^{X,\boldsymbol{j}} \equiv \mathrm{e}^{t\mathcal{L}^{X,\boldsymbol{j}}}$ to denote the corresponding semigroup, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{P}_{t-s}^{X,\boldsymbol{j}}\partial_{\boldsymbol{j}}\boldsymbol{P}_s^X\boldsymbol{f} = \boldsymbol{P}_{t-s}^{X,\boldsymbol{j}}(-\mathcal{L}^{X,\boldsymbol{j}}\partial_{\boldsymbol{j}} + \partial_{\boldsymbol{j}}\mathcal{L}^X)\boldsymbol{P}_s^X\boldsymbol{f} = -|X|\boldsymbol{P}_{t-s}^{X,\boldsymbol{j}}(\partial_{\boldsymbol{j}})\boldsymbol{P}_s^X\boldsymbol{f}$$
$$+ \sum_{\boldsymbol{k}:\boldsymbol{j}\notin X+\boldsymbol{k}} \boldsymbol{P}_{t-s}^{X,\boldsymbol{j}}([\partial_{\boldsymbol{j}}, \mathcal{L}_{X+\boldsymbol{k}}]\boldsymbol{P}_s^X\boldsymbol{f}) + \sum_{Y:Y\ni\boldsymbol{j}} \boldsymbol{P}_{t-s}^{X,\boldsymbol{j}}([\partial_{\boldsymbol{j}}, \delta_{\Psi_Y}]\boldsymbol{P}_s^X\boldsymbol{f}).$$

Hence we obtain

$$||\partial_{\boldsymbol{j}}\boldsymbol{P}_t^X\boldsymbol{f}|| \leqslant \mathrm{e}^{-t|X|}||\partial_{\boldsymbol{j}}\boldsymbol{f}|| + \int_0^t \mathrm{d}s\, \mathrm{e}^{-(t-s)|X|} \sum_{\boldsymbol{k}:\boldsymbol{j}\notin X+\boldsymbol{k}} ||[\partial_{\boldsymbol{j}}, \mathcal{L}_{X+\boldsymbol{k}}]\boldsymbol{P}_s^X\boldsymbol{f}||$$
$$+ \int_0^t \mathrm{d}s\, \mathrm{e}^{-(t-s)|X|}|\lambda| \cdot \sum_{Y:Y\ni\boldsymbol{j}} ||[\partial_{\boldsymbol{j}}, \delta_{\Psi_Y}]\boldsymbol{P}_s^X\boldsymbol{f}||. \tag{2.1}$$

Now we observe that

$$\sum_{\boldsymbol{k}:\boldsymbol{j}\notin X+\boldsymbol{k}} ||[\partial_{\boldsymbol{j}}, \mathcal{L}_{X+\boldsymbol{k}}]\boldsymbol{P}_s^X\boldsymbol{f}|| \leqslant \sum_{\boldsymbol{k}:\boldsymbol{j}\notin X+\boldsymbol{k}} \sum_{\boldsymbol{i}\in X+\boldsymbol{k}} \tilde{\eta}_{\boldsymbol{ki}}^{\boldsymbol{j}} ||\partial_{\boldsymbol{i}}\boldsymbol{P}_s^X\boldsymbol{f}|| \tag{2.2}$$

with some constants $\tilde{\eta}_{\boldsymbol{k}\mathrm{i}}^{\boldsymbol{j}} \geqslant 0$, and

$$\sum_{Y:Y\ni\boldsymbol{j}} ||[\partial_{\boldsymbol{j}}, \delta_{\Psi_Y}]\boldsymbol{P}_s^X\boldsymbol{f}|| \leqslant \sum_{Y:Y\ni\boldsymbol{j}} ||\Psi_Y||\{2||\partial_{\boldsymbol{j}}\boldsymbol{P}_s^X\boldsymbol{f}|| + \sum_{\boldsymbol{k}\in Y} ||\partial_{\boldsymbol{k}}\boldsymbol{P}_s^X\boldsymbol{f}||\}. \tag{2.3}$$

Inserting equations (2.2) and (2.3) into equation (2.1), after the summation over $\boldsymbol{j} \in \mathbb{Z}^d$ we arrive at the following bound

$$|||\boldsymbol{P}_t^X\boldsymbol{f}||| \leqslant \mathrm{e}^{-t|X|}|||\boldsymbol{f}||| + (\tilde{\varepsilon}|X| + |\lambda|(2||\Psi||_1 + ||\Psi||_2)) \int_0^t \mathrm{d}s \mathrm{e}^{-(t-s)|X|}|||\boldsymbol{P}_s^X\boldsymbol{f}||| \tag{2.4}$$

with some constant $\tilde{\varepsilon}$ dependent on ε and $\eta(\cdot)$. If the assumptions of the theorem are satisfied with sufficiently small $\varepsilon_0 > 0$ and $\lambda_0 > |\lambda| > 0$, using the inequality equation (2.4) one easily obtains the exponential decay with

$$m = (1-\tilde{\varepsilon})|X| - |\lambda| \cdot (2||\Psi||_1 + ||\Psi||_2). \qquad \square$$

If we choose generalized conditional expectations and the generator of the Hamiltonian dynamics to be associated to the same potential, we know that the family of invariant states contains all KMS states associated to a given potential at a given temperature. If we additionally omit the generator of the Hamiltonian dynamics, we even obtain the detailed balance condition (in our symmetric L_2 space).

Some simple perturbation arguments suggest that the conditions of the above theorem are true for generalized conditional expectations discussed in [7]. Under these conditions we also have strong control on the approximation of the infinite volume semigroup $\boldsymbol{P}_t^X$ by

$\boldsymbol{P}_t^{\Lambda,X}$ semigroups. The second part of the above theorem implies the uniqueness of the invariant state for the dynamics $\boldsymbol{P}_t^X$ and the spectral gap in the $\mathbb{L}_2(\varphi_1, \frac{1}{2})$-spectrum of the corresponding generator. Finally, let us remark that in the classical case [3], one concludes that the existence of spectral gap for all the other spin flip dynamics $\boldsymbol{P}_t^Y$, $Y \subset\subset \mathbb{Z}^d$, $\lambda = 0$. This is due to the fact that the corresponding quadratic forms of the generators $\mathcal{L}^Y$ are all mutually equivalent. One may expect that a similar property should also remain true in the noncommutative context.

In the next section we provide a class of nontrivial examples for which we can verify all the conditions used above. These examples are constructed by a proper extension procedure applied to the classical spin systems. Therefore it should be no surprise that we can recover for them many features of the classical case including the Feller property existence and strong ergodicity, as well as the results on the spectral gap for all dynamics with equivalent Dirichlet forms (similar to [3]).

The new feature of our examples, not present for the case of classical discrete spin systems, is the fact that we can have the Hamiltonian term in the generator. In the appendix we show that it does not commute with the purely dissipative one (and thus our jump type dynamics do not belong to the class of semigroups subordinated to the Hamiltonian automorphism). It is clear that this class will serve as a valuable laboratory for the further research in this domain.

3. A class of examples

Let M be the single spin algebra consisting of $n \times n$ matrices. Let M_c be a single spin space of cardinality n for a classical spin system. For a given representation of M, we can identify the set of diagonal elements diag(M) in M with the (complex-valued) functions on M_c. (This procedure depends on the choice of the representation of the matrices and therefore it can be done in many ways.) Let ι denote the inverse of this map. It is clear that the inductive limit algebra $\mathcal{B}$ over diag(M), called later the classical subalgebra, can be identified with the set $\mathcal{C}$ of continuous functions on the space $\Omega_c \equiv M_c^{\mathbb{Z}^d}$ of configurations of a classical spin system. For $\boldsymbol{f} \in \mathcal{B}$ its unique correspondent in $\mathcal{C}$ will be denoted by f_c. In this setting we have that for any $\boldsymbol{f} \in \mathcal{B}$

$$\mathrm{Tr}\,\boldsymbol{f} = \mu_0 f_c$$

where Tr and μ_0 denote the normalized trace and the free measure (defined on Ω_c as the product of uniform probability measures on M_c), respectively. A unique (for a fixed isomorphism ι) potential given by $\{\iota(\Phi_X) \in \mathcal{B}\}_{X \subset\subset \mathbb{Z}^d}$ corresponding to a potential $\Phi \in \mathbb{B}_1$ will be called a classical potential. Later (with a slight abuse of the notation) we will use the same symbol to denote the classical potential for the quantum spin system and its counterpart for the commutative spin system.

Let μ_Φ be a Gibbs measure on Ω_c corresponding to the potential $\Phi \equiv \{\Phi_X \in \mathcal{C}\}_{X \subset\subset \mathbb{Z}^d}$. For a σ-algebra Σ_Λ generated by classical spins in a finite set $\Lambda \subset\subset \mathbb{Z}^d$ we define a density matrix

$$\rho_\Phi^{(\Lambda)} \equiv \frac{\mathrm{d}\mu_\Phi|\Sigma_\Lambda}{\mathrm{d}\mu_0|\Sigma_\Lambda}.$$

Setting $\rho^{(\Lambda)} \equiv \iota(\rho_\Phi^{(\Lambda)}) \in \mathcal{B}$ to be the correspondent of $\rho_\Phi^{(\Lambda)}$ we can define an infinite volume state

$$\omega_\Phi(\boldsymbol{f}) \equiv \lim_{\mathcal{F}_0} \mathrm{Tr}(\rho^{(\Lambda)}\boldsymbol{f}).$$

Clearly on the elements of the classical subalgebra $\boldsymbol{f} \in \mathcal{B}$ we have

$$\omega_\Phi(\boldsymbol{f}) = \mu_\Phi(f_c).$$

We can also introduce the Hamiltonian dynamics

$$\alpha_t^\Phi(\boldsymbol{f}) \equiv \lim_{\mathcal{F}_0} (\rho^{(\Lambda)})^{-\mathrm{i}t} \boldsymbol{f} (\rho^{(\Lambda)})^{\mathrm{i}t}.$$

One can see that the limit depends on the potential Φ but is independent of the Gibbs measure chosen. Moreover the classical subalgebra is pointwise invariant with respect to this Hamiltonian dynamics.

Following the construction of [7] we can introduce the interpolating family $\mathbb{L}_p(\omega_\Phi, \frac{1}{2})$, $p \in [1, \infty]$. One can note that they coincide with the classical L_p spaces associated to the Gibbs measure μ_Φ when restricted to the classical subalgebra. In the symmetric L_2 space we have the following result.

Proposition. For every potential $\Phi \in \mathbb{B}_1$ and any $X \subset\subset \mathbb{Z}^d$ the following generalized conditional expectation is well defined

$$E_X^\Phi(\boldsymbol{f}) = \mathrm{Tr}_X\, \gamma_X^* \boldsymbol{f} \gamma_X$$

with

$$\gamma_X = \gamma_X^* \equiv \mathrm{e}^{-\frac{1}{2}U_X} (\mathrm{Tr}_X\, \mathrm{e}^{-U_X})^{-\frac{1}{2}} \in \mathcal{B} \subset \mathcal{A}$$

where $U_X \equiv \sum_{Y \cap X \neq \emptyset} \boldsymbol{\Phi}_Y$. Moreover E_X^Φ is symmetric in $\mathbb{L}_2(\omega_\Phi, \frac{1}{2})$ and if the potential Φ is of finite range we have $E_X^\Phi(\mathcal{A}_0) \subseteq \mathcal{A}_0$.

Note that on the classical subalgebra $\mathcal{B}$ the generalized conditional expectation E_X introduced in this proposition coincides with the conditional expectation of a Gibbs state for the same potential. Using the results described in section 2 we obtain the following.

Theorem. Let $\Phi, \Psi \in \mathbb{B}_2$ and assume that Φ is a classical potential. Let $\mathcal{L}_\Phi^{X,\Lambda}$ be a finite volume jump-type generator corresponding to the generalized conditional expectations E_{X+j}^Φ, for a given finite set $X \subset\subset \mathbb{Z}^d$ and $j \in \Lambda \subset\subset \mathbb{Z}^d$, and let $\boldsymbol{P}_t^{X,\Phi,\Psi,\Lambda} \equiv \mathrm{e}^{t(\mathcal{L}_\Phi^{X,\Lambda} + \lambda\delta_\Psi)}$. Then the infinite semigroup

$$\boldsymbol{P}_t^{X,\Phi,\Psi} \equiv \lim_{\mathcal{F}_0} \boldsymbol{P}_t^{X,\Phi,\Psi,\Lambda}$$

is well defined unit and positivity-preserving semigroup which has the Feller property, that is

$$\boldsymbol{P}_t^{X,\Phi,\Psi}(\mathcal{A}) \subseteq \mathcal{A}.$$

The extension of the classical spin system gives us a possibility to introduce dissipative dynamics with nontrivial Hamiltonian term. In case when Ψ is also a classical potential the Hamiltonian term cannot be detected on the classical subalgebra. Moreover if $\Phi = \Psi$ the set of invariant states contains all the Gibbs states (in general we can have many of them). We would like to stress that even in this case the semigroup with Hamiltonian term is nontrivial; as we show in the appendix the operator $\mathcal{L}_X$ does not commute with the Hamiltonian semigroup.

4. Ergodicity

Let $\beta\Phi = \{\beta\Phi_X\}_{X\subset\subset\mathbb{Z}^d}$ be a classical potential of finite range. Applying the general theory described in section 2 we have the following ergodicity result.

Theorem. Given a finite set $X \subset\subset \mathbb{Z}^d$, there is $\beta_0 > 0$ and $\lambda_0 > 0$ such that for any $|\beta| < \beta_0$ and $|\lambda| < \lambda_0$ we have

$$|||\boldsymbol{P}_t^{X,\beta\Phi,\lambda\Psi}(\boldsymbol{f})||| \leqslant \mathrm{e}^{-mt}|||\boldsymbol{f}|||$$

with a constant $m \in (0, \infty)$ independent of $\boldsymbol{f}$.

By this we have got an explicit class of dissipative dynamics for quantum spin systems on a lattice with strong ergodic properties. In general if we choose $\Phi \neq \Psi$, we may not know *a priori* the unique invariant state. It is known that in the case of commutative spin systems of classical statistical mechanics one can get ergodicity for the larger range of temperatures when choosing the elementary generators $\mathcal{L}_{X+j}$ to be defined with a cube X of large size, see [3]. In the commutative situation there is a standard way of using the strong ergodicity result for such dynamics (without Hamiltonian term), to prove the $\mathbb{L}_2$ ergodicity of all the other block spin-flip generators. The main idea of the proof of this fact is based on the equivalence of Dirichlet forms associated to generators $\mathcal{L}^Y$ with $Y \subset\subset \mathbb{Z}^d$. Below we prove that this equivalence remains true in the considered case of quantum block spin flip generators associated to finite-range potentials.

Theorem. For every $X, Y \subset\subset \mathbb{Z}^d$ there is a constant $C_{X,Y} \in (0, \infty)$ such that

$$C_{X,Y}^{-1}\langle \boldsymbol{f}, -\mathcal{L}_\Phi^X \boldsymbol{f}\rangle_{\mathbb{L}_2(\omega_\Phi,\frac{1}{2})} \leqslant \langle \boldsymbol{f}, -\mathcal{L}_\Phi^Y \boldsymbol{f}\rangle_{\mathbb{L}_2(\omega_\Phi,\frac{1}{2})} \leqslant C_{X,Y}\langle \boldsymbol{f}, -\mathcal{L}_\Phi^X \boldsymbol{f}\rangle_{\mathbb{L}_2(\omega_\Phi,\frac{1}{2})}.$$

Therefore, if for some $X \subset\subset \mathbb{Z}^d$ the corresponding dynamics $\mathrm{e}^{t\mathcal{L}_\Phi^X}$ is strongly exponentially ergodic, then for any $Y \subset\subset \mathbb{Z}^d$ the corresponding block spin-flip generator $\mathcal{L}_\Phi^Y$ as a self-adjoint operator in $\mathbb{L}_2(\omega_\Phi, \frac{1}{2})$ has a spectral gap and the corresponding stochastic dynamics is $\mathbb{L}_2$-ergodic.

Proof. It is sufficient to show the equivalence of the Dirichlet forms in the case when $X = \{0\}$, i.e. equivalence of any block spin-flip generator to the single spin-flip one. It is clear that one can reduce our problem to proper estimates on elementary generators. We will need the following proposition in which we use the following notation

$$\partial_R Y = \{\boldsymbol{k} \in \mathbb{Z}^d \setminus Y : \forall \boldsymbol{j} \in Y |\boldsymbol{k} - \boldsymbol{j}| \leqslant R\}.$$

To simplify the notation, later we will set $\langle\cdot,\cdot\rangle \equiv \langle\cdot,\cdot\rangle_{\mathbb{L}_2(\omega_\Phi,\frac{1}{2})}$ and $||\cdot||_2 \equiv ||\cdot||_{\mathbb{L}_2(\omega_\Phi,\frac{1}{2})}$.

Proposition 4.1. Let Φ be a classical potential of finite range R. Then for $Z \equiv Y \cup \partial_R Y$, we have

$$\langle \boldsymbol{f}, -\mathcal{L}_Y \boldsymbol{f}\rangle \leqslant C_Z \sum_{\boldsymbol{i}\in Z} ||\partial_{\boldsymbol{i}} \boldsymbol{f}||_2^2 \tag{4.1}$$

where C_Z is a positive constant depending on Z while $\partial_{\boldsymbol{i}}\boldsymbol{f} \equiv \boldsymbol{f} - \mathrm{Tr}_{\{\boldsymbol{i}\}}\,\boldsymbol{f}$.

Moreover, there is a constant $C > 0$ such that for any $\boldsymbol{i} \in \mathbb{Z}^d$ we have

$$||\partial_{\boldsymbol{i}}\boldsymbol{f}||_2^2 \leqslant C\langle \boldsymbol{f}, -\mathcal{L}_{\{\boldsymbol{i}\}}\boldsymbol{f}\rangle. \tag{4.2}$$

Applying this proposition to $\mathcal{L}_{Y+j}$, by summation over j it is easy to obtain the inequality

$$\langle f, -\mathcal{L}^Y f\rangle \leqslant C_{Y,0}\langle f, -\mathcal{L}^0 f\rangle.$$

Proof of proposition 4.1. Let us first observe that for $Y \cup \partial_R Y \subset Z$ we have

$$E_Y \operatorname{Tr}_Z f = \operatorname{Tr}_Y \gamma_Y^* \operatorname{Tr}_Z f \gamma_Y = \operatorname{Tr}_Z f \operatorname{Tr}_Y \gamma_Y^* \gamma_Y = \operatorname{Tr}_Z f \tag{4.3}$$

where we have used the fact that $\gamma_Y = \mathrm{e}^{-\frac{1}{2}U_Y}(\operatorname{Tr}_Y \mathrm{e}^{-U_Y})^{-\frac{1}{2}}$ commutes with any operator localized in $\mathbb{Z}^d \setminus \{Y \cup \partial_R Y\}$. Hence $\mathcal{L}_Y \operatorname{Tr}_Z f = 0$ and we have

$$\langle f, -\mathcal{L}_Y f\rangle = \langle f, -\mathcal{L}_Y(f - \operatorname{Tr}_Z f)\rangle = \langle -\mathcal{L}_Y f, (f - \operatorname{Tr}_Z f)\rangle \leqslant \frac{a}{2}|| - \mathcal{L}_Y f||_2^2$$
$$+\frac{1}{2a}||(f - \operatorname{Tr}_Z f)||_2^2 \leqslant \frac{a}{2}||\mathcal{L}_Y|| \cdot \langle f, -\mathcal{L}_Y f\rangle + \frac{1}{2a}||(f - \operatorname{Tr}_Z f)||_2^2 \tag{4.4}$$

for any $a \in (0, \infty)$; we have used here the fact $\mathcal{L}_Y$ is a symmetric operator on the Hilbert space $\mathbb{L}_2$ with a bounded norm $||\mathcal{L}_Y|| \equiv ||\mathcal{L}_Y||_{\mathbb{L}_2 \to \mathbb{L}_2}$ (cf [7]). Choosing $||\mathcal{L}_Y|| < 2/a$ we arrive at

$$\langle f, -\mathcal{L}_Y f\rangle \leqslant C_1||(f - \operatorname{Tr}_Z f)||_2^2$$

with $C_1 \equiv a^{-1} \cdot (2 - a||\mathcal{L}_Y||)^{-1}$. Now we observe

$$||f - \operatorname{Tr}_Z f||_2^2 = |Z|^2 || \frac{1}{|Z|} \sum_{i_n \in Z} (\operatorname{Tr}_{\{k<i_n\}} f - \operatorname{Tr}_{\{k \leqslant i_n\}} f)||_2^2$$
$$\leqslant |Z| \sum_{i_n \in Z} || \operatorname{Tr}_{\{k<i_n\}} f - \operatorname{Tr}_{\{k \leqslant i_n\}} f||_2^2 \tag{4.5}$$

where in the last step the Holder inequality was used; $|Z|$ denotes the number of sites in Z and we assumed a convention $f \equiv \operatorname{Tr}_{\{\emptyset\}} f$. To complete the proof of the first part of proposition 4.1 we need to observe the following fact.

Lemma 4.2. Under the above assumptions,

$$|| \operatorname{Tr}_{\{l\}} f||_2 \leqslant || \operatorname{Tr}_{\{l\}} || \cdot ||f||_2. \tag{4.6}$$

Proof of lemma 4.2. It is sufficient to prove the bound of interest to us for finite volume states, because by the appropriate limiting procedure the infinite volume bound follows. Let $\mathbb{L}_2(\omega, s)$, $s \in [0, 1]$, be the one-parameter family of interpolating Hilbert spaces introduced in [7], section 5, (see also [6], section 2). We recall that the scalar product of $\mathbb{L}_2(\omega, s)$ is given by

$$\langle f, f\rangle_{\omega_\varrho, s} = \operatorname{Tr}(\varrho^{(1-s)} f^* \varrho^s f). \tag{4.7}$$

(In particular, $|| \cdot ||_2 \equiv || \cdot ||_{\mathbb{L}_2(\omega, \frac{1}{2})}$.) We have

$$\omega(| \operatorname{Tr}_{\{l\}} f|^2) \leqslant \omega(\operatorname{Tr}_{\{l\}} |f|^2) \leqslant || \operatorname{Tr}_{\{l\}} || \cdot \omega(|f|^2) \tag{4.8}$$

and

$$\omega(| \operatorname{Tr}_{\{l\}} f^*|^2) \leqslant \omega(\operatorname{Tr}_{\{l\}} |f^*|^2) \leqslant || \operatorname{Tr}_{\{l\}} || \cdot \omega(|f^*|^2) \tag{4.9}$$

where the Kadison–Schwarz inequality for $\operatorname{Tr}_{\{l\}}(\cdot)$ was used. Then (4.6) follows from the interpolation procedure based on the three lines theorem. □

Now, applying lemma 4.2 to inequality (4.5) we get

$$\begin{aligned} ||\boldsymbol{f} - \mathrm{Tr}_Z\, \boldsymbol{f}||_2^2 &\leqslant |Z| \sum_{\boldsymbol{i}_n \in Z} ||\,\mathrm{Tr}_{\{\boldsymbol{k}<\boldsymbol{i}_n\}}\,||^2 ||\boldsymbol{f} - \mathrm{Tr}_{\{\boldsymbol{i}_n\}}\, \boldsymbol{f}||_2^2 \\ &\leqslant |Z|||\,\mathrm{Tr}_{\{\boldsymbol{l}\}}\,||^{2|Z|} \sum_{\boldsymbol{i}\in Z} ||\boldsymbol{f} - \mathrm{Tr}_{\{\boldsymbol{i}\}}\, \boldsymbol{f}||_2^2 \leqslant |Z|||\,\mathrm{Tr}_{\{\boldsymbol{l}\}}\,||^{2|Z|} \sum_{\boldsymbol{i}\in Z} ||\partial_{\{\boldsymbol{i}\}} \boldsymbol{f}||_2^2. \end{aligned} \tag{4.10}$$

Consequently, equations (4.10) and (4.4) lead to

$$\langle \boldsymbol{f}, -\mathcal{L}_Y \boldsymbol{f}\rangle \leqslant C_1 |Z| \cdot ||\,\mathrm{Tr}_{\{\boldsymbol{l}\}}\,||^{2|Z|} \sum_{\boldsymbol{i}\in Z} ||\partial_{\{\boldsymbol{i}\}} \boldsymbol{f}||_2^2. \tag{4.11}$$

This completes the proof of the first part of proposition 4.1. □

The second part follows from the following lemma.

Lemma 4.3. Under the conditions stated above,

$$||\boldsymbol{f} - \mathrm{Tr}_{\{\boldsymbol{i}\}}\, \boldsymbol{f}||_2^2 \leqslant C_2 \langle \boldsymbol{f}, -\mathcal{L}_{\{\boldsymbol{i}\}} \boldsymbol{f}\rangle \tag{4.12}$$

where C_2 is a positive constant.

Proof. The proof is based on the following simple observation.

$$\mathrm{Tr}_{\{\boldsymbol{i}\}}\, E_{\{\boldsymbol{i}\}} \boldsymbol{f} = \mathrm{Tr}_{\{\boldsymbol{i}\}} (\mathrm{Tr}_{\{\boldsymbol{i}\}}\, \gamma^*_{\{\boldsymbol{i}\}} \boldsymbol{f} \gamma_{\{\boldsymbol{i}\}}) = \mathrm{Tr}_{\{\boldsymbol{i}\}}\, \mathrm{Tr}_{\{\boldsymbol{i}\}}\, \gamma^*_{\{\boldsymbol{i}\}} \boldsymbol{f} \gamma_{\{\boldsymbol{i}\}} = \mathrm{Tr}_{\{\boldsymbol{i}\}}\, \gamma^*_{\{\boldsymbol{i}\}} \boldsymbol{f} \gamma_{\{\boldsymbol{i}\}} = E_{\{\boldsymbol{i}\}} \boldsymbol{f}. \tag{4.13}$$

So

$$\partial_{\{\boldsymbol{i}\}} E_{\{\boldsymbol{i}\}} \boldsymbol{f} = 0.$$

Therefore, using lemma 4.2, we obtain

$$\begin{aligned} ||\partial_{\{\boldsymbol{i}\}} \boldsymbol{f}||_2^2 &= ||\partial_{\{\boldsymbol{i}\}} (\boldsymbol{f} - E_{\{\boldsymbol{i}\}} \boldsymbol{f})||_2^2 \equiv ||\partial_{\{\boldsymbol{i}\}} (\mathcal{L}_{\{\boldsymbol{i}\}} \boldsymbol{f})||_2^2 \leqslant (1 + ||\,\mathrm{Tr}_{\{\boldsymbol{i}\}}\,||)^2 \cdot ||\mathcal{L}_{\{\boldsymbol{i}\}} \boldsymbol{f}||_2^2 \\ &\leqslant (1 + ||\,\mathrm{Tr}_{\{\boldsymbol{i}\}}\,||)^2 \cdot ||\mathcal{L}_{\{\boldsymbol{i}\}}|| \langle \boldsymbol{f}, -\mathcal{L}_{\{\boldsymbol{i}\}} \boldsymbol{f}\rangle. \end{aligned} \tag{4.14}$$

This completes the proof of lemma 4.3 and thus the proof of proposition 4.1 also. □

To prove the lower bound of interest to us it is sufficient to prove the following result.

Proposition 4.4. Let Y be a finite region of the lattice $\mathbb{Z}^d$ and let $\{\boldsymbol{i}\} \subset Y$. Then

$$||\partial_{\{\boldsymbol{i}\}} \boldsymbol{f}||_2^2 \leqslant C_3 \langle \boldsymbol{f}, -\mathcal{L}_Y \boldsymbol{f}\rangle \tag{4.15}$$

where C_3 is a positive constant independent of i. Moreover we have

$$\langle \boldsymbol{f}, -\mathcal{L}_{\{\boldsymbol{i}\}} \boldsymbol{f}\rangle \leqslant C_4 \sum_{\boldsymbol{j}\in\{\boldsymbol{i}\}\cup\partial_R\{\boldsymbol{i}\}} \langle \boldsymbol{f}, -\mathcal{L}_{Y+\boldsymbol{j}} \boldsymbol{f}\rangle \tag{4.16}$$

for any $Y \subset\subset \mathbb{Z}^d \ni 0$ with some constant $C_4 \in (0, \infty)$ independent of $\boldsymbol{i}$.

Proof. For any $Y \subset\subset \mathbb{Z}^d \ni 0$ we have

$$\begin{aligned} ||\partial_{\{\boldsymbol{j}\}} \boldsymbol{f}||_2^2 &= ||\partial_{\{\boldsymbol{j}\}} (\mathcal{L}_{Y+\boldsymbol{j}} \boldsymbol{f})||_2^2 \leqslant (1 + ||\,\mathrm{Tr}_{\{\boldsymbol{j}\}}\,||)^2 ||\mathcal{L}_{Y+\boldsymbol{j}} \boldsymbol{f}||_2^2 \\ &\leqslant (1 + ||\,\mathrm{Tr}_{\{\boldsymbol{j}\}}\,||)^2 \cdot ||\mathcal{L}_{Y+\boldsymbol{j}}|| \cdot \langle \boldsymbol{f}, -\mathcal{L}_Y \boldsymbol{f}\rangle. \end{aligned} \tag{4.17}$$

This completes the proof of the first part of proposition 4.4. Now, applying the first part of Proposition 4.1 with the one point set $\{\boldsymbol{i}\}$ and $Z = \{\boldsymbol{i}\} \cup \partial_R\{\boldsymbol{i}\}$, we obtain

$$\langle \boldsymbol{f}, -\mathcal{L}_{\{\boldsymbol{i}\}} \boldsymbol{f}\rangle \leqslant C_Z \sum_{\boldsymbol{j}\in\{\boldsymbol{i}\}\cup\partial_R\{\boldsymbol{i}\}} ||\partial_{\boldsymbol{j}} \boldsymbol{f}||_2^2 \tag{4.18}$$

and by the first part of proposition 4.4. we arrive at

$$\langle \boldsymbol{f}, -\mathcal{L}_{\{i\}}\boldsymbol{f}\rangle \leqslant C_4 \sum_{\boldsymbol{j}\in\{\boldsymbol{i}\}\cup\partial_R\{\boldsymbol{i}\}} \langle \boldsymbol{f}, -\mathcal{L}_{Y+\boldsymbol{j}}\boldsymbol{f}\rangle \tag{4.19}$$

with some non-negative constant C_4. □

Summing inequality (4.16) over $\boldsymbol{i} \in \mathbb{Z}^d$ we arrive at

$$\langle \boldsymbol{f}, -\mathcal{L}^0\boldsymbol{f}\rangle \leqslant C_{Y,0}\langle \boldsymbol{f}, -\mathcal{L}^Y\boldsymbol{f}\rangle. \tag{4.20}$$

This together with the inequality after proposition 4.1 prove the equivalence of Dirichlet forms of $\mathcal{L}^Y$ and $\mathcal{L}^0$ for any finite set $Y \subset\subset \mathbb{Z}^d$. From this the general case easily follows and the proof of equivalence of Dirichlet forms of all block spin-flip generators is finished. □

Appendix A

Proposition. Let $\Phi \in \mathbb{B}_2$ be a classical potential and let $\mathcal{L}_{X,\Phi}$ be the jump-type generator corresponding to the generalized conditional expectations E^{Φ}_{X+j} for a given finite set $X \subset\subset \mathbb{Z}^d$ and $j \in \mathbb{Z}^d$. Then $\mathcal{L}_{X,\Phi}$ does not commute with δ_Φ and therefore the corresponding infinite semigroup $\mathrm{e}^{t\mathcal{L}_{X,\Phi}}$ is not subordinated to the Hamiltonian dynamics α^{Φ}_t.

Proof. Let us suppose *a contrario* that the Markov dynamics $\mathrm{e}^{\mathcal{L}_{X,\Phi}}$ commutes with α^{Φ}_t. Then

$$\sum_Y [\Phi(Y), \mathrm{Tr}_X\, \gamma^*_X f \gamma_X] = \sum_Y \mathrm{Tr}_X\, \gamma^*_X[\Phi(Y), f]\gamma_X \tag{A.1}$$

for any $f \in \mathcal{A}_0(\Lambda)$. It is worth pointing out that $\gamma^*_X = \gamma_X$ as it is a function on the set $\{\sigma^i_z\}$ of self-adjoint and commuting elements. Writing the interaction Φ in the form

$$\Phi(Y) = \sum_Z c(Y, Z)\sigma_z^{X\cap Z}\sigma_z^{Z\setminus X}$$

where $c(Y, Z)$ are numbers, the left-hand side of (A.1) can be rewritten as

$$\sum_Y\sum_Z c(Y, Z)[\sigma_z^{X\cap Z}\sigma_z^{Z\setminus X}, \mathrm{Tr}_X\, \gamma^*_X f \gamma_X] = \sum_Y\sum_Z c(Y, Z)\sigma_z^{X\cap Z}[\sigma_z^{Z\setminus X}, \mathrm{Tr}_X\, \gamma^*_X f \gamma_X]. \tag{A.2}$$

The right-hand side of (A.1) takes the form

$$\begin{aligned}\sum_Y\sum_Z c(Y, Z)\, \mathrm{Tr}_X\, \gamma^*_X[\sigma_z^{X\cap Z}\sigma_z^{Z\setminus X}, f]\gamma_X &= \sum_Y\sum_Z c(Y, Z)\, \mathrm{Tr}_X\, \sigma_z^{X\cap Z}\gamma^*_X[\sigma_z^{Z\setminus X}, f]\gamma_X \\ &\quad + \sum_Y\sum_Z c(Y, Z)\{\mathrm{Tr}_X\, \gamma^*_X[\sigma_z^{X\cap Z}, f]\gamma_X\}\sigma_z^{Z\setminus X}.\end{aligned} \tag{A.3}$$

It is an easy observation that (A.2) depends on spins in $X \cap Z$ while (A.3) is independent of the operators. Therefore, the equality (A.1) cannot be true. Consequently, our assumption on commutativity of α^{Φ}_t with $\mathrm{e}^{t\mathcal{L}_{X,\Phi}}$ is false and the proof is complete. □

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